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Theoretical and numerical analysis of large-scale heat transfer problems with temperature-dependent pore-fluid densities

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Abstract

Purpose – In many scientific and engineering fields, large-scale heat transfer problems with temperature-dependent pore-fluid densities are commonly encountered. For example, heat transfer from the mantle into the upper crust of the Earth is a typical problem of them. The main purpose of this paper is to develop and present a new combined methodology to solve large-scale heat transfer problems with temperature-dependent pore-fluid densities in the lithosphere and crust scales.

Design/methodology/approach – The theoretical approach is used to determine the thickness and the related thermal boundary conditions of the continental crust on the lithospheric scale, so that some important information can be provided accurately for establishing a numerical model of the crustal scale. The numerical approach is then used to simulate the detailed structures and complicated geometries of the continental crust on the crustal scale. The main advantage in using the proposed combination method of the theoretical and numerical approaches is that if the thermal distribution in the crust is of the primary interest, the use of a reasonable numerical model on the crustal scale can result in a significant reduction in computer efforts.

Findings – From the ore body formation and mineralization points of view, the present analytical and numerical solutions have demonstrated that the conductive-and-advective lithosphere with variable pore-fluid density is the most favorite lithosphere because it may result in the thinnest lithosphere so that the temperature at the near surface of the crust can be hot enough to generate the

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shallow ore deposits there. The upward throughflow (i.e. mantle mass flux) can have a significant effect on the thermal structure within the lithosphere. In addition, the emplacement of hot materials from the mantle may further reduce the thickness of the lithosphere.

Originality/value – The present analytical solutions can be used to: validate numerical methods for solving large-scale heat transfer problems; provide correct thermal boundary conditions for numerically solving ore body formation and mineralization problems on the crustal scale; and investigate the fundamental issues related to thermal distributions within the lithosphere. The proposed finite element analysis can be effectively used to consider the geometrical and material complexities of large-scale heat transfer problems with temperature-dependent fluid densities.

Keywords Numerical analysis, Boundary layers, Heat transfer

Paper type Research paper

1. Introduction

Over the past several decades, computer methods have been found more and more applications to geological problems. For example, on the mantle scale, numerical methods were widely used to simulate convective flow and heat transfer in the mantle of the Earth (Richter, 1973; Buck and Parmentier, 1986; Doin *et al.*, 1997). On the crustal scale, numerical methods were developed and extensively used to solve ore body formation and mineralization problems within the upper crust of the Earth (Yeh and Tripathi, 1989; Phillips, 1991; Nield and Bejan, 1992; Raffensperger and Garven, 1995a, b; Zhao *et al.*, 1997, 1998, 1999, 2000, 2001, 2002, 2003). Since the ore body forming process is strictly associated with material deformation, pore-fluid flow, heat transfer, mass transport and related chemical reactions, sophisticated numerical methods were developed to solve the fully coupled problem between the material deformation, pore-fluid flow, heat transfer, mass transport and related chemical reactions in hydrothermal systems within the crust of the Earth. Since the thickness of the mantle (i.e. a few thousands of kilometers) is much greater than that of the crust (i.e. only a few tens of kilometers), the crust acts as the skin of a huge body in the whole crust-mantle system. From the computational science point of view, the large differences in thickness between the mantle and the crust create a severe difficulty, when the whole crust-mantle system is modeled simultaneously in a computer simulation. For instance, the finite element mesh designed to produce a useful solution for the mantle cannot give any meaningful solution to the crust, because the mesh scale is too large to model the crustal details. On the other hand, the finite element mesh designed to simulate the detailed phenomena within the crust may become computationally impractical because a huge number of degree-of-freedom are created to model the mantle.

To overcome this difficulty, the current numerical practice is to simulate the crust and mantle separately. If the behavior of the mantle is of interest, an effective mesh is used to model the mantle, while the crust is treated as an up boundary in the numerical model. However, if the detailed phenomenon within the crust is of interest, an effective mesh is used to model the crust, while the crust-mantle interface is treated as a bottom boundary in the numerical analysis. Since the crust-mantle interface (i.e. the Moho) is located in the lithosphere of the Earth, it is important to understand the detailed thermal structure within the lithosphere so that the boundary condition at the crust-mantle interface can be reasonably determined for a numerical model of the crustal scale. For this reason, the scientific questions we need to answer are: given a conductive heat flux from the sub-lithospheric mantle, what is the fundamental mechanism of mass and heat transport from the continental mantle into the continental upper crust? And if mass and heat transportation from the continental mantle to the continental upper crust is significant, what are the stable thickness and the related thermal structure within the continental lithosphere during the corresponding thermal event? The answers to these questions are scientifically significant, at least, for the following two points. First, if the mantle conductive heat flux at the bottom of the continental lithosphere is known, the temperature distribution within the continental lithosphere can be determined mathematically from the related scientific principles. As a result, the bottom of the continental lithosphere, at which the temperature is about 1,500 K by the geological definition, can be identified so that the stable lithosphere thickness can be determined. This constraint can be used to investigate the thermal thinning phenomenon of the continental lithosphere during a thermal event in the mantle of the Earth. Second, if the continental lithosphere thickness is known, the mantle conductive heat flux during the formation of the continental lithosphere can be deduced from the relationship between the lithosphere thickness and the mantle conductive heat flux. Therefore, the temperature at the crust-mantle interface can be evaluated from the analytical solution on the lithospheric scale, so that both the crustal thickness and the thermal boundary condition at the crust-mantle interface can be determined for this particular thermal event. This provides the necessary conditions for establishing a numerical model of the crustal scale. The resulting numerical model of the crustal scale, with the related numerical methods such as the finite element method and finite difference method, can be used to investigate the detailed ore body forming processes in the upper crust of the Earth during a thermal event in the upper part of the mantle.

Since the problem to be considered in this study can be, in essence, attributed to a large-scale heat transfer problem in porous media with temperature-dependent pore-fluid densities, a combination of using theoretical and numerical approaches is proposed to solve the problem. That is to say, the theoretical approach is used to determine the thickness and the related thermal boundary conditions of the continental crust on the lithospheric scale, whereas the numerical approach is used to simulate the detailed structures and complicated geometries of the continental crust on the crustal scale. This means that we may take advantages of both analytical and numerical approaches to deal with the large-scale heat transfer problems in porous media using two different scales. On the large scale (i.e. the lithospheric scale), the detailed structures of relatively small scale within the whole problem domain may be neglected so that the problem can be solved using the pure mathematical approach. The resulting theoretical solution is then used to determine the thickness and the related thermal boundary conditions of the continental crust, which are the necessary conditions for establishing a numerical model of the crustal scale. However, on the relatively small scale (i.e. the crustal scale), the detailed structures and complicated geometries of the continental crust can be simulated in the numerical model. The main advantage in using the proposed combination method is that if the thermal structure in the crust is of the primary interest, the use of a numerical model on the crustal scale can result in a significant reduction in computer efforts. For this particular reason, the major purpose of this paper is to derive analytical solutions for the relationship between the continental lithosphere thickness and the mantle conductive heat flux on the lithospheric scale.

To establish the relationship between the continental lithosphere thickness and the mantle conductive heat flux at the bottom of the lithosphere during a particular thermal event is an old geological problem. The traditional geological way to find a solution for this problem is to use the one-dimensional heat conduction equation in the lithosphere without considering the fluid flow leaked from the lithospheric mantle of the Earth. The solution derived by this assumption suffers many difficulties in explaining several geological phenomena occurring in the crust of the Earth. For example, without considering the upward flow through the continental crust, the temperature distribution predicted by the pure conductive models is not enough to explain ore deposits in some hydrothermal systems in the upper crust, although these ore deposits have already existed. On the other hand, the extensive research during the past decades has indicated that due to the high temperature and low strength of the lithospheric mantle, the permeability of the lithospheric mantle is very small (i.e. in the level of about $10^{-18}\,m^2$ or less) so that the pressure-gradient of the pore-fluid is equal to the lithostatic pressure-gradient in the lower crust and lithospheric mantle of the Earth. This requires an upward flow to exist within the continental lithosphere, which is leaked from the lithospheric mantle and lower crust. The recent research in the field of geochemistry has demonstrated that the amount of the pore-fluid, which is required to maintain the pore-fluid pressure gradient to be lithostatic in the continental lithosphere, may be generated by dehydration and devolatilization reactions occurring in the lower crust of the Earth. As it will be addressed in Section 2 of this paper, the recently developed porosity wave concept provides a useful tool for explaining the possible transport process of mass and heat from the continental mantle into the upper crust of the Earth. Therefore, it is possible to use these recent concepts to revisit the relationship between the continental lithosphere thickness and the mantle conductive heat flux within the continental lithosphere during a particular thermal event. Keeping this in mind, the upward throughflow of the mantle material in the lithosphere and the temperature-dependent density of the pore-fluid are considered to derive a new relationship between the continental lithosphere thickness and the mantle conductive heat flux within the lithosphere during a particular thermal event. The analytical solutions developed can be also used to validate any numerical methods for solving lithosphere-scale heat transfer problems with temperature-dependent pore-fluid densities.

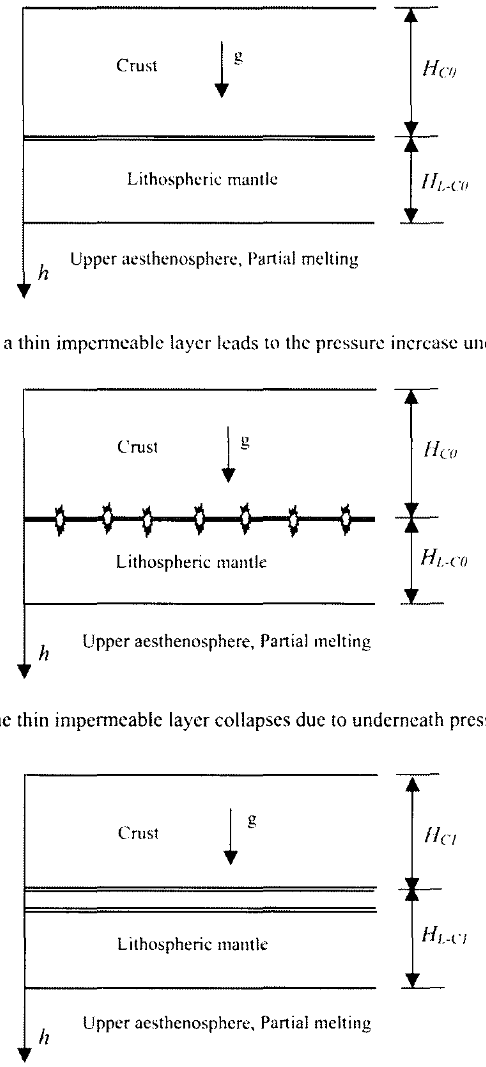
2. Possible mechanism for material exchange between the mantle and the crust

Material exchange between the continental mantle and the continental crust of the Earth may cause a significant change in the thermal structure of the continental lithosphere. The continuous replacement of the crust material by the mantle material can reduce the overall thickness of the lithosphere. If the heat flux generated by the material exchange between the continental mantle and the continental crust is transferred into the upper crust, it may create an appropriate environment for ore body formation and mineralization to take place in the near surface of the Earth. This process may become an efficient mechanism to generate near-surface ore deposits. However, since the mantle material can be treated as a viscous magma and the crust material can be treated as a porous medium, there is a mechanism question on the material exchange between the continental mantle and the continental crust of the

Earth. To answer this question scientifically, a new conceptual model for material exchange between the continental mantle and the continental crust is proposed. The basic idea behind the proposed conceptual model is that the porosity wave, which was originally proposed for dealing with both magma ascending in the lithospheric mantle and upward pore-fluid flow in sedimentary basins (Richter and Mckenzie, 1984; Mckenzie, 1984, 1987; Barcilon and Richter, 1986; Connolly and Podladchikov, 1998, 2000), can also transport materials from the continental mantle into the continental crust through the two key processes: magma solidification and porous material consolidation. In what follows, a conceptual model is proposed to show how the mantle material is transported into the upper crust of the Earth through the generation and propagation of the porosity wave in the continental lithosphere of the Earth.

As shown in Figure 1, the continental lithosphere is composed of porous materials, although the porosity may be very small. Note that in this figure, g is the gravity acceleration. From the structural geology point of view, the whole lithosphere can be divided into crust and lithospheric mantle by the Moho. Since the upper aesthenosphere of the Earth is in a partial melting state, it is possible to transport the magma within the lithospheric mantle through the generation and propagation of the porosity wave (Richter and Mckenzie, 1984; Mckenzie, 1984, 1987). Recent studies (refer Barnes (1997) and the related references therein) also demonstrated that the solidification temperature of the upward propagating magma can be significantly reduced if water exists in the lithospheric mantle. For instance, the solidification temperature of the water-bearing magma may be reduced to about 1,100 K in the lithosphere. This means that the upward propagating magma may be transported to the Moho through the generation and propagation of the porosity wave in the lithospheric mantle. As a result, the porous material of the lithospheric mantle (i.e. between the Moho and the top of the aesthenosphere of the Earth) may be filled with the upward propagating magma, which originally comes from the aesthenosphere of the Earth. On the other hand, the porous material of the crust (i.e. between the Moho and the surface of the Earth) is filled with pore-fluids, which may be either gas or liquid, depending on the local temperature and pressure conditions. Thus, the Moho surface may be assumed to be a relatively impermeable thin layer due to the solidification of the upward propagating magma. This implies that the Moho surface may be considered as a solidification surface of the ascending magma. The initial thickness of the continental crust is H_{C0}, while the initial thickness of the continental lithosphere is H_{C0} plus H_{L-C0}. Thus, the initial thickness of the lithospheric mantle is H_{L-C0}. Owing to an increase of the pressure caused by the gravity, the porosity of the porous material decreases with the depth from the surface of the Earth. Since the strength of the porous material is temperature dependent, it also decreases with the depth due to the existence of the downward positive temperature gradient in the lithosphere.

At the initial stage ($t = t_0$), any perturbation of either temperature or pressure at the bottom of the continental lithosphere due to mantle convection or mantle crust interaction can result in magma ascending from the aesthenosphere into the lithospheric mantle. The ascending magma moves toward the Moho surface through the generation and propagation of the porosity wave (Richter and Mckenzie, 1984; Mckenzie, 1984, 1987). Since the Moho surface is assumed to be a relatively impermeable thin layer, the upward propagating magma accumulates just under it until the increased magma pressure due to magma accumulation exceeds the material



($t=t_0$: Moho of a thin impermeable layer leads to the pressure increase underneath the layer)

($t=t_1$: The thin impermeable layer collapses due to underneath pressure increase)

($t=t_2$: Formation of a new upward thin layer leads to the rise of Moho, while formation of a new downward thin layer results in downward movement of crustal material)

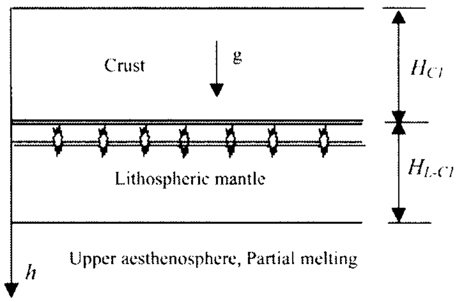
Figure 1.
Sketch of transport processes of continental crust and mantle materials through porosity waves

strength of this thin layer. Generally, the increased magma pressure due to the magma accumulation might play two important roles in the generation of the porosity wave near the Moho surface. First, the increased magma pressure can cause an expansion of pores (or create pores if they do not exist) so that the porosity just underneath the Moho surface is increased. Secondly, the increased magma pressure can cause the collapse of the impermeable thin layer since the material strength of this layer is relatively very low due to its location in a rather high temperature region. Once the increased magma pressure exceeds the material strength of the impermeable thin layer, the accumulated magma outbursts and penetrates the thin layer. This is the first stage of the porosity wave generation near the Moho surface ($t = t_1$). The outbursted magma can travel upward some distance due to the local extra pressure gradient created by the magma accumulation at the initial stage until it becomes solidified due to heat loss to the surrounding matrix. The solidification of the ascending outbursted magma can generate a new upward impermeable thin layer above the initial Moho surface. This consequence is equivalent to the upward movement of the initial Moho surface. Also, the released volatiles during magma solidification can travel upward in the crust in exactly the same form of the porosity wave as pore-fluid travels upward in sedimentary basins (Barcilon and Richter, 1986; Mckenzie, 1987; Connolly and Podladchikov, 2000). This implies that the mantle volatile material can also be transported into the upper crust by the porosity waves. At the same time, the expanded pores of the underlying material of the initial impermeable layer may become consolidated and closed due to the release of the local pressure. As a result, a new downward impermeable layer is generated under the initial Moho surface. This consequence is equivalent to the downward movement of the initial crust material. The generation of the new upward and downward impermeable layers marks the propagation of the generated porosity wave. This may be considered as the second stage of the generation and propagation of the porosity wave near the Moho surface ($t = t_2$). As these processes repeat and continuously progress (see $t = t_3$, t_4 and t_5 in Figure 2), the formation of the second new upward thin impermeable layer leads to the further rise of the Moho surface, while the formation of the third new downward thin impermeable layer results in the downward movement of more crustal materials. This indicates that as the porosity wave travels upwards, the mantle material in the form of either magma or volatile pore-fluid moves upwards in the continental lithosphere, while the crust material moves downwards in the lithosphere of the Earth. This implies that it is possible to transport mass and heat from the continental mantle to upper crust through the generation and propagation of the porosity wave in the continental lithosphere of the Earth.

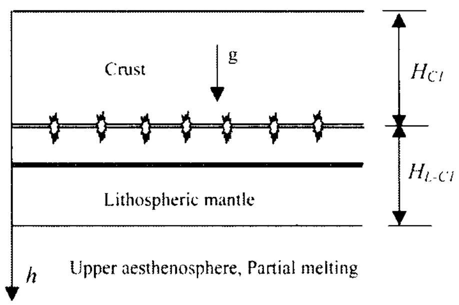
As the downwardly moved crust material replaces the initial lithospheric mantle material, the replaced lithospheric mantle material may cause the rest of the lithospheric mantle material to move downwardly so that some of the lithospheric mantle material at the bottom of the lithosphere may be melted and the whole thickness of the continental lithosphere is gradually reduced. On the other hand, the upward movement of the Moho surface can lead to the thinning of the continental crust of the Earth. Clearly, the adjustment of the thermal structure in the continental lithosphere due to the upward mass and heat fluxes carried by the porosity wave may be an important mechanism to reduce the thickness of the continental lithosphere. In order to understand this mechanism, theoretical solutions for the heat transfer problem with the temperature-dependent pore-fluid density on the lithospheric scale will be derived mathematically to investigate the effects of the upward propagating mass and heat fluxes on the thermal structure of the continental lithosphere.

3. Derivation of analytical solutions for large-scale heat transfer problems with temperature-dependent fluid densities

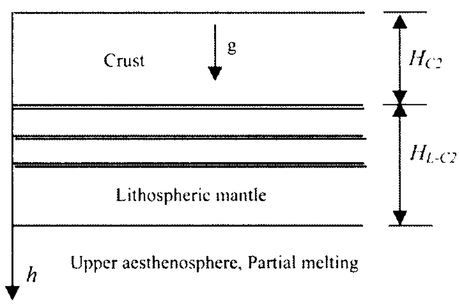
On the lithosphere scale, the problem can be treated as a one-dimensional model in the vertical direction. Once the continental lithosphere of the Earth is in a stable state, heat



($t=t_3$: The new downward thin impermeable layer collapses due to underneath pressure increase)



($t=t_4$: The collapse of the new upward thin layer leads to magma ascending, while formation of the second new downward thin layer results in further downward movement of crustal material)



($t=t_5$: Formation of the second new upward thin layer leads to the further rise of Moho, while formation of the third new downward thin layer results in downward movement of more crustal material)

Figure 2. Sketch of transport processes of continental crust and mantle materials through porosity waves

transfer within the lithosphere reaches a thermodynamic equilibrium state, which can be regarded as a quasisteady state. As shown in Figure 3, if the temperature-dependent density of the pore-fluid in the lithosphere and the upward throughflow, which may represent the mass flux from the lithospheric mantle into upper crust of the Earth, are considered, the governing equation for the steady-state heat transfer in the continental lithosphere can be expressed as follows.

$$\frac{\mathrm{d}}{\mathrm{d}y}(v\rho_f c_{pf} T) - \frac{\mathrm{d}}{\mathrm{d}y}\left(\lambda^e \frac{\mathrm{d}T}{\mathrm{d}y}\right) = 0 \qquad (1)$$

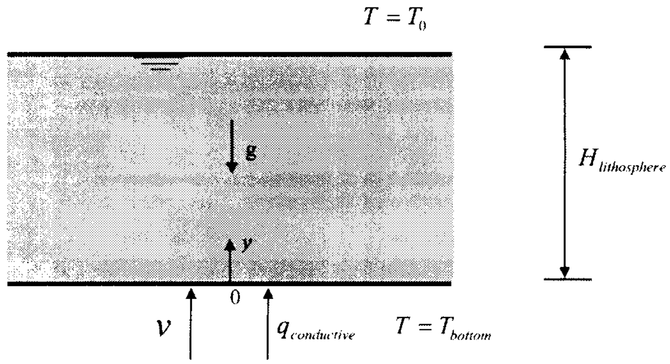


Figure 3.
Definition of heat transfer problem in the continental lithosphere

where ρ_f is the density of the pore-fluid; c_{pf} is the specific heat of the pore-fluid; v is the upward throughflow velocity in the lithosphere; T is the temperature of the lithosphere material; and λ^e is the thermal conductivity coefficient of the lithosphere material.

It is noted that the heat source/sink term due to heat generation within the continental lithosphere is neglected in equation (1). In this case, equation (1) states that for any given point in the continental lithosphere, the variation in the sum of the advective heat flux and conductive heat flux is equal to zero.

From the continuum mechanics point of view, the temperature-dependent density of the pore-fluid and the thermal conductive coefficient of the lithosphere material can be written as:

$$\rho_f = \rho_0[1 - \beta(T - T_0)] \tag{2}$$

$$\lambda^e = \phi\lambda + (1 - \phi)\lambda^s \tag{3}$$

where ρ_0 and T_0 are the reference density of the pore-fluid and the reference temperature of the medium; λ and λ^s are the thermal conductivity coefficients for the pore-fluid and solid matrix in the lithothsphere; and ϕ and β are the porosity of the lithosphere material and the thermal volume expansion coefficient of the pore-fluid.

Substituting equation (2) into equation (1) yields the following equation.

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{v(1+\beta T_0)}{D}T - \frac{v\beta}{D}T^2 - \frac{\mathrm{d}T}{\mathrm{d}y}\right] = 0 \tag{4}$$

where D is the thermal diffusivity coefficient which is expressed as

$$D = \frac{\lambda^e}{\rho_0 c_{pf}} \tag{5}$$

Clearly, equation (4) has a solution as follows.

$$\frac{\mathrm{d}T}{\mathrm{d}y} - \frac{v(1+\beta T_0)}{D}T + \frac{v\beta}{D}T^2 = C_1 \tag{6}$$

where C_1 is a constant. This constant can be determined using the heat flux continuum condition at the bottom of the lithosphere and expressed as follows.

$$C_1 = -\frac{q_{\text{conductive}}}{\lambda^e} - \frac{v(1+\beta T_0)}{D} T_{\text{bottom}} + \frac{v\beta}{D} T_{\text{bottom}}^2 \tag{7}$$

where $q_{\text{conductive}}$ is the mantle conductive heat flux transferred from the aesthenosphere to the lithosphere of the Earth; and T_{bottom} is the temperature at the bottom of the lithosphere.

If the upward throughflow velocity in the lithosphere is constant, equation (6) can be rewritten as:

$$\frac{\mathrm{d}T}{\mathrm{d}y} - aT + bT^2 = C_1 \tag{8}$$

where

$$a = \frac{v(1+\beta T_0)}{D}, \quad b = \frac{v\beta}{D} \tag{9}$$

The homogeneous equation of equation (8) is

$$\frac{\mathrm{d}T}{\mathrm{d}y} - aT + bT^2 = 0 \tag{10}$$

A general solution to equation (10) can be derived and expressed as:

$$T_{\text{general}} = \frac{a}{C_2 \mathrm{e}^{-ay} + b} \tag{11}$$

where C_2 is a constant to be determined.

Also, the particular solution to equation (8) can be derived and expressed as:

$$T_{\text{particular}} = \frac{\frac{a}{b}\left(1 - \sqrt{1 + \frac{4bC_1}{a^2}}\right)}{2} \tag{12}$$

The physical meaning of this particular solution is that if the whole lithosphere could be in an isothermal state, then the temperature in the whole lithosphere would be equal to the temperature expressed by this particular solution. Therefore, this particular solution represents the extent to which the heat flux is transferred from the aesthenosphere to the lithosphere of the Earth.

Note that the condition, under which equation (12) holds, is expressed as

$$\frac{q_{\text{conductive}}}{v} \leq \frac{\lambda^e}{4D\beta}\{(1+\beta T_0)^2 - 4\beta T_{\text{bottom}}[1 - \beta(T_{\text{bottom}} - T_0)]\} \tag{13}$$

Figure 4 shows the constraint condition between the upward throughflow velocity and the mantle conductive heat flux, where the unit for the velocity and heat flux is m/s and W/m^2, respectively. This condition indicates that when the density of the pore-fluid in the lithosphere is temperature-dependent, the thermal structure cannot reach a steady state unless this constraint condition is satisfied within the lithosphere. In this regard, the related lithosphere is called the conditionally stable lithosphere.

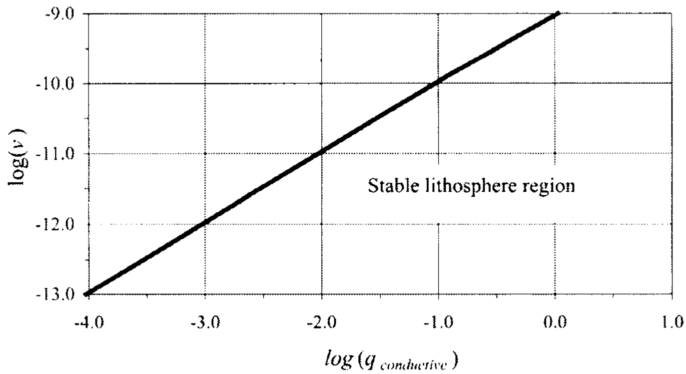


Figure 4.
Constraint condition between the velocity and conductive heat flux

Adding equations (11) and (12) together yields the solution to equation (8) as follows.

$$T = \frac{a}{C_2 e^{-ay} + b} + \frac{\frac{a}{b}\left(1 - \sqrt{1 + \frac{4bC_1}{a^2}}\right)}{2} \tag{14}$$

From the geological definition, the temperature at the bottom of the lithosphere is $T_{\text{bottom}} \approx 1,500\,\text{K}$. This boundary condition (i.e. $T = T_{\text{bottom}}$ at $y = 0$) can be used to determine the second constant C_2. This leads to the following result.

$$C_2 = \frac{a}{T_{\text{bottom}} - T_{\text{particular}}} - b \tag{15}$$

The top surface boundary condition (i.e. $T = T_0$ at $y = H_{\text{lithosphere}}$) is used to yield an expression for the thickness of the lithosphere as follows.

$$H_{\text{lithosphere}} = -\frac{1}{a} \ln\left[\frac{1}{C_2}\left(\frac{a}{T_0 - T_{\text{particular}}} - b\right)\right] \tag{16}$$

Substituting the related constants into equation (16) results in the following equation:

$$H_{\text{lithosphere}} = \frac{D}{v(1+\beta T_0)} \ln\left[\frac{\left(\frac{1+\beta T_0}{\beta} - T_{\text{bottom}} + T_{\text{particular}}\right)(T_0 - T_{\text{particular}})}{\left(\frac{1+\beta T_0}{\beta} - T_0 + T_{\text{particular}}\right)(T_{\text{bottom}} - T_{\text{particular}})}\right] \tag{17}$$

Since $T_{\text{particular}}$ is a function of both the pore-fluid velocity and conductive heat flux from the aesthenosphere to the lithosphere, equation (17) expresses the relationship between the continental lithosphere thickness and the mantle conductive heat flux within the lithosphere during a particular thermal event. Due to the consideration of the temperature-dependent density of the pore-fluid, the lithosphere considered is called the heat-conductive-and-advective lithosphere with a pore-fluid of variable density.

In the case of no upward throughflow (i.e. $v = 0$) within the continental lithosphere, the relationship between the lithosphere thickness and the mantle conductive heat flux within the lithosphere during a particular thermal event can be derived and expressed as follows.

$$\hat{H}_{\text{lithosphere}} = \frac{\lambda^e (T_{\text{bottom}} - T_0)}{q_{\text{conductive}}} \tag{18}$$

Since heat conduction is the only heat transfer mechanism in this situation, the lithosphere considered here is called the heat-conductive-only lithosphere. The corresponding solution for the temperature distribution in this situation is:

$$\hat{T} = T_{\text{bottom}} - \frac{q_{\text{conductive}}}{\lambda^e} y \tag{19}$$

Another special case is when β is equal to zero in equation (4). This means that the density of the pore-fluid is constant within the lithosphere. Therefore, the lithosphere considered in this particular situation is called the heat-conductive-and-advective lithosphere with the constant density of the pore-fluid. Under this condition, the solution for the temperature distribution can be derived and expressed as:

$$\tilde{T} = -\frac{q_{\text{conducive}} D}{v \lambda^e} e^{\frac{v}{D} y} + \frac{q_{\text{conductive}} D}{v \lambda^e} + T_{\text{bottom}} \tag{20}$$

The corresponding lithosphere thickness in this case is:

$$\tilde{H}_{\text{lithosphere}} = \frac{D}{v} \ln \left[\frac{v \lambda^e}{q_{\text{conductive}} D} (T_{\text{bottom}} - T_0) + 1 \right] \tag{21}$$

4. Numerical modeling of large-scale heat transfer problems with temperature-dependent fluid densities

Although the analytical solutions derived in the previous section is useful for the fundamental understanding of heat transfer in a homogenous lithosphere of the Earth, the heterogeneity and two-dimensional nature of the lithosphere are neglected in the process of deriving the analytical solutions so that any unexpected mathematical difficulties can be avoided. However, for the purpose of considering the heterogeneity and two-dimensional nature of the continental lithosphere, the numerical method such as the finite element method is useful to generate numerical solutions. Since the finite element method is an approximate method, the numerical solution obtained from the finite element modeling needs to be validated when it is used to deal with large-scale heat transfer problems with temperature-dependent fluid densities.

For a two-dimensional heat transfer problem with temperature-dependent fluid densities, equation (1) in the previous section can be extended into the following form.

$$\frac{\partial}{\partial x}(u \rho_f c_{pf} T) + \frac{\partial}{\partial y}(v \rho_f c_{pf} T) - \left[\frac{\partial}{\partial x}\left(\lambda^e \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y}\left(\lambda^e \frac{\partial T}{\partial y} \right) \right] = 0 \tag{22}$$

where ρ_f is the density of the pore-fluid; c_{pf} is the specific heat of the pore-fluid; u is the horizontal pore-fluid velocity in the lithosphere; v is the upward throughflow velocity of the pore-fluid in the lithosphere; T is the temperature of the lithosphere material; and λ^e is the thermal conductivity coefficient of the lithosphere material.

Insertion of equation (2) into equation (22) yields the following equation.

$$[\rho_0 c_{pf}(1+\beta T_0)]\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)-2\beta\rho_0 c_{pf}T\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)$$
$$-\lambda^e\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right]=0 \tag{23}$$

It is noted that equation (23) is a non-linear equation of strong non-linearity, which is clearly expressed by the term,

$$T\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right),$$

in the above equation. Due to this strong non-linearity, it is impossible to obtain an analytical solution for equation (23). Therefore, the finite element method (Zienkiewicz, 1977) is used to solve the above equation in this study.

Following the conventional procedure of a finite element analysis, the discretized finite element formulation of equation (23) can be expressed as follows.

$$(\rho_0 c_{pf}(1+\beta T_0)[B]-2\beta\rho_0 c_{pf}[C]+[H])\{T\}=\{Q\} \tag{24}$$

where $[B]$, $[C]$ and $[H]$ are global property matrices; $\{T\}$ is the global vector of nodal temperature; and $\{Q\}$ is the global vector of the nodal "load". These global matrices and vectors can be determined by assembling all the corresponding elemental matrices and vectors, which are of the following definitions.

$$[H_e]=\iint_A\left(\lambda^e\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x}+\lambda^e\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\right)\mathrm{d}A \tag{25}$$

$$[B_e]=\iint_A\left(u[N]^T\frac{\partial[N]}{\partial x}+v[N]^T\frac{\partial[N]}{\partial y}\right)\mathrm{d}A \tag{26}$$

$$[C_e]=\iint_A\left(T_{j-1}u[N]^T\frac{\partial[N]}{\partial x}+T_{j-1}v[N]^T\frac{\partial[N]}{\partial y}\right)\mathrm{d}A \tag{27}$$

$$\{Q_e\}=-\int_S[N]^T(q_x n_x+q_y n_y)\,\mathrm{d}S \tag{28}$$

where $[H_e]$, $[B_e]$ and $[C_e]$ are the property matrices of a finite element; $\{Q_e\}$ the "load" vector of the element; $[N]$ the shape function matrix of the element; q_x and q_y are the heat flux in the x and y directions; A is the area of the element; S is the boundary of the element; n_x and n_y are the direction cosines of the outward unit normal vector on the boundary of the element.

It is noted that in order to evaluate the elemental property matrix, $[C_e]$, the temperature in the previous iteration step is required. This means that the resulting finite element equation must be solved in an iteration manner. Although many different kinds of iteration methods are available for solving non-linear equations, the successive substitution iteration method is used in this study due to its simplicity and easy to be implemented in the finite element computation. However, the use of an iteration method implies that an initial guess of the nodal temperature vector, $\{T_0\}$, has to be available at the beginning of the finite element computation. For the large-scale heat transfer in the lithosphere, the geothermal temperature distribution under the pure heat conduction condition is an ideal candidate for the initial guess of the nodal temperature vector, $\{T_0\}$. As a result, there is no hurdle to be overcome in the forthcoming finite element analysis of large-scale heat transfer problems with temperature-dependent fluid densities.

5. Application examples

Analytical solutions are very important for scientific and engineering problems (Zhao and Steven, 1996). For example, an analytical solution can be used as a powerful tool to gain an understanding of the solution scenarios under some extreme conditions for a given problem. In addition, an analytical solution is often a useful or even in some circumstances a unique measure in the assessment and validation of any numerical methods. Therefore, the analytical solutions for different kinds of benchmark problems often play an important role in developing standards for the assessment of the correctness and credibility of modeling and simulation in scientific and engineering computations. In this particular study, the present analytical solution can be used to determine the thickness of the lithosphere if the mantle conductive heat flux at the bottom of the lithosphere is given. Alternatively, the present analytical solution can also be used to determine the correct thermal boundary conditions, which is crucial to establish a numerical model of the crustal scale, at the bottom of the crust if the thickness of the lithosphere is known.

5.1 Determination of boundary conditions of a numerical model of lithospheric scale

In order to construct a numerical model of the lithospheric scale, we need to determine the model size in the depth direction and the related thermal boundary conditions. This can be achieved in the following two ways. In the first way, one can prescribe the temperature boundary condition at the bottom of the model so that the depth of the model has to be determined using the analytical solutions derived in Section 3. Alternatively, in the second way, one can prescribe the depth of the model so that the temperature boundary condition at the bottom of the model has to be determined using the analytical solutions. Thus, we need a theoretical curve to express the relationship between the lithosphere thickness and the mantle conductive heat flux. In order to produce such a curve, we need some basic parameters of the continental lithosphere. For this purpose, the following parameters are used in the forthcoming analysis. For pore-fluid, reference density is 1,000 kg/m^3; volumetric thermal expansion coefficient is $2.07 \times 10^{-4}(1/°C)$; specific heat is 4,000 J/(kg °C). For the lithosphere material, thermal conductivity coefficient is 2.25 W/(m °C); specific heat is 815 J/(kg °C). Temperature at the top and bottom of the lithosphere is 25°C and 1,225°C, respectively.

Figure 5 shows the relationship between the lithosphere thickness and the mantle conductive heat flux due to four different upward throughflow velocities within the lithosphere of constant pore-fluid density. It needs to be pointed out that the curve with $v = 0$ represents the results for the heat-conductive-only lithosphere, while the curves in correspondence with $v = 2 \times 10^{-12}$, 2×10^{-11} and 2×10^{-10} m/s represent the results for the heat-conductive-and-advective lithosphere. Since the pore-fluid density is constant, the lithosphere is unconditionally thermodynamic stable. This means that for any set of given mantle conductive heat flux and upward throughflow velocity, a steady-state lithosphere can be maintained, from the thermodynamic point of view. Clearly, with an increase in the upward throughflow velocity, there is a significant decrease in the lithosphere thickness. Since the upward throughflow velocity and the mantle conductive heat flux may be deduced from the surface geological measurements and observations, the lithosphere thickness can be straightforwardly determined from the curves shown in this figure.

If the pore-fluid density within the lithosphere varies strongly with temperature, the lithosphere becomes conditionally stable. In this case, the stable lithosphere can only be maintained when the mantle conductive heat flux and upward throughflow velocity satisfy the constraint condition expressed in equation (13). Figure 6 shows the relationship between the lithosphere thickness and the mantle conductive heat flux in the case of $v = 2 \times 10^{-11}$ m/s within the lithosphere of constant and temperature-dependent pore-fluid densities. It is obvious that consideration of the temperature-dependent pore-fluid density can cause a significant reduction in the lithosphere thickness, compared with that of the constant pore-fluid density. This recognition may have an important geological implication for the ore body formation and mineralization in hydrothermal systems within the upper crust of the Earth.

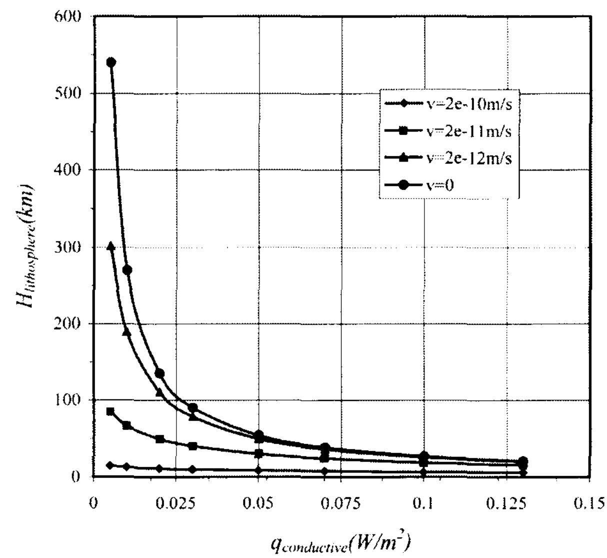


Figure 5.
Effect of pore-fluid velocity on the thickness of continental lithosphere

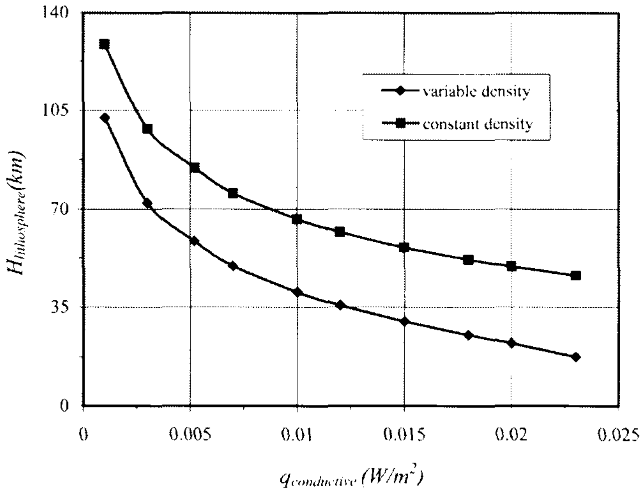


Figure 6.
Effect of pore-fluid density on the thickness of continental lithosphere ($v = 2 \times 10^{-11}$ m/s)

5.2 Validation and application of numerical modeling of large-scale heat transfer problems with temperature-dependent fluid densities

In this sub-section, we first demonstrate how to use the theoretical curve of the lithosphere thickness versus the mantle conductive heat flux to determine the depth of a numerical model and the related thermal boundary condition. For example, for a given conductive heat flux, we can use the theoretical curve to find the thickness of the lithosphere, depending on the upward throughflow velocity within the lithosphere. Then we can use the following equation to determine the temperature at the bottom of the numerical model.

$$T_{\text{bottom}} = T_0 + \frac{Hq_{\text{conductive}}}{\lambda^e} \tag{29}$$

where H can be $H_{\text{lithosphere}}$, $\hat{H}_{\text{lithosphere}}$, or $\tilde{H}_{\text{lithosphere}}$, depending on the upward throughflow velocity within the lithosphere.

In order to validate the present finite element formulation of large-scale heat transfer problems with temperature-dependent fluid densities, three different kinds of lithospheres with different thermal regimes, namely the conductive-only lithosphere, the conductive-and-advective lithosphere with constant pore-fluid density (i.e. cases with $v = 2 \times 10^{-12}$ and 2×10^{-11} m/s) and the conductive-and-advective lithosphere with variable (i.e. temperature-dependent) pore-fluid density, are considered in the forthcoming finite element computations. If the mantle conductive heat flux is chosen as 0.02 W/m^2 for all the three kinds of lithospheres, the depth of the numerical model (i.e. the thickness of the lithosphere) can be determined using the procedure described previously. The lithosphere thickness is 135 km for the conductive-only lithosphere, while it is 22.14 km for the conductive-and-advective lithosphere with variable pore-fluid density. These two lithospheres may be regarded as two extreme cases. For the intermediate cases, namely the conductive-and-advective lithosphere with constant pore-fluid density, the lithosphere thickness is 110.3 km in the case of $v = 2 \times 10^{-12}$ m/s, but it is 49.4 km in the case of $v = 2 \times 10^{-11}$ m/s.

Once the depth of the numerical model is determined, the lithosphere is simulated using the finite element method. Figure 7 shows the finite element mesh of the numerical model. In this figure, 5,790 quadrilateral 6-node triangular elements are used to represent the whole computational domain. The length of the computational domain is fixed to be 50 km, while the depth of the computational domain can have different values, depending on the upward throughflow velocity within the lithosphere. The basic parameters used in the finite element computation are exactly the same as those used in the theoretical analysis of the lithosphere in Section 5.1. The temperature at the bottom of the model is determined to be 1,225°C for all the four numerical models considered here.

Figure 8 shows the temperature distribution due to three different kinds of lithospheres with different thermal regimes, namely the conductive-only lithosphere, the conductive-and-advective lithosphere with constant pore-fluid density (i.e. cases with $v = 2 \times 10^{-12}$ and 2×10^{-11} m/s) and the conductive-and-advective lithosphere with variable (i.e. temperature-dependent) pore-fluid density. In this figure, diamonds stand for the numerical solution for the conductive-only lithosphere, while crosses stand for the numerical solution for the conductive-and-advective lithosphere with variable (i.e. temperature-dependent) pore-fluid density. For the conductive-and-advective lithosphere with constant pore-fluid density (i.e. cases with $v = 2 \times 10^{-12}$ and 2×10^{-11} m/s) in the figure, triangles and squares stand for the numerical results in the case of $v = 2 \times 10^{-12}$ and 2×10^{-11} m/s, respectively. The analytical solutions for all the four cases are shown by the corresponding solid lines. It is obvious that the numerical solutions agree very well with the analytical ones for all the four situations. This demonstrates that the present finite element formulation is valid for dealing with large-scale heat transfer problems with temperature-dependent fluid densities. In addition, both the numerical and analytical solutions indicate that the upward throughflow can significantly reduce the resulting lithosphere thickness. This recognition may have some considerable implications in geology. Generally, for a given thermal event of the Earth, the conductive-and-advective lithosphere with variable pore-fluid density results in the hottest upper crust, while the conductive-only lithosphere results in the coldest upper crust, from the thermodynamic equilibrium point of view. Since the upward throughflow may have a significant effect on the thermal structure in the lithosphere, great cautions must be taken when the nature of the pore-fluid is considered within the lithosphere.

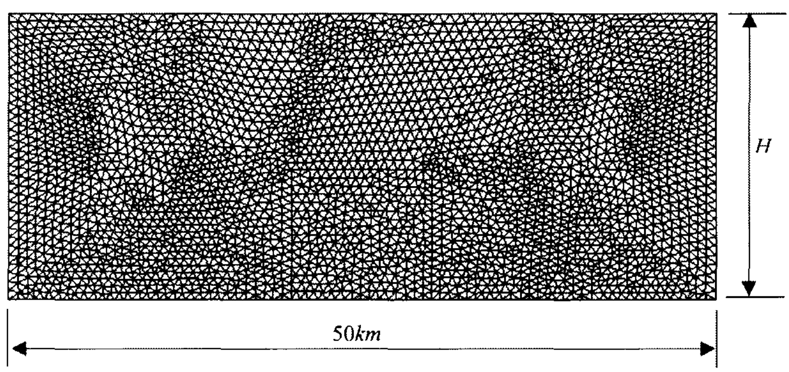


Figure 7.
Finite element mesh of the benchmark problem

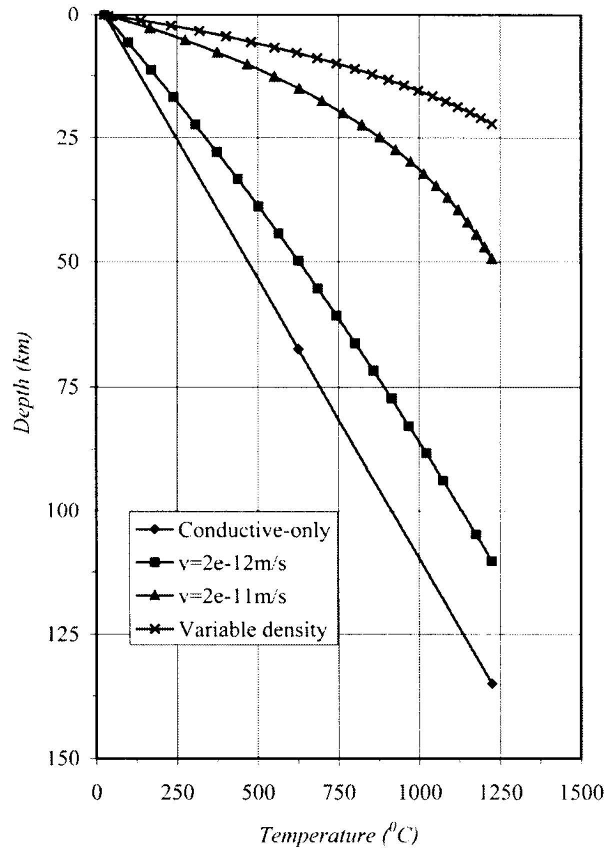


Figure 8.
Temperature distribution due to different lithosphere

After the present finite element formulation is validated through the above benchmark problems, it can be used to investigate the effect of material heterogeneity on the thermal structure of the lithosphere. For this purpose, a more thermal conductive region representing a magma emplacement is considered in the centre bottom of the computational model. The thermal conductivity of this particular region is ten times that of the remaining regions of the computational model. For the sake of saving space, only the afore-mentioned conductive-and-advective lithosphere with variable (i.e. temperature-dependent) pore-fluid density is considered in this situation. The size of the more thermal conductive region is assumed to be 10 km wide and 10 km high in the finite element computation.

Figure 9 shows the temperature contours of the computational model including the heterogeneity effect of material thermal conductivity. Since heat transfer proceeds more efficiently in the region of a relatively high thermal conductivity, the temperature in this particular region is relatively higher than that in the corresponding other region, so that the isothermal line is no longer a horizontal line near the region of a relatively high thermal conductivity, compared with the case of the whole lithosphere of

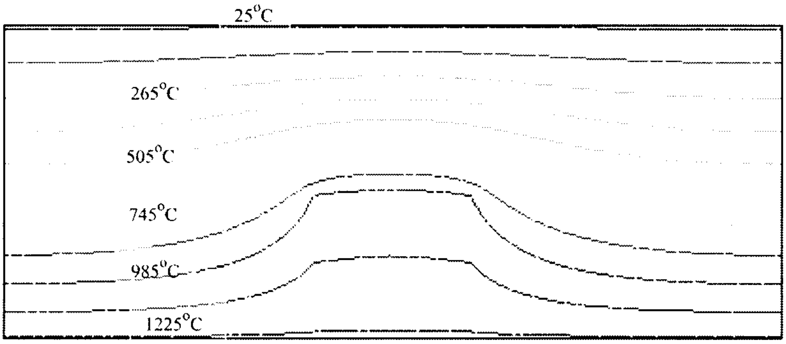


Figure 9.
Temperature distribution due to heterogeneity of heat conductivity

homogeneous thermal conductivity. This implies that the emplacement of hot materials from the mantle may further reduce the thickness of the lithosphere.

6. Conclusions

Some important issues related to the theoretical and numerical computations of temperature distributions in large-scale heat transfer problems with temperature-dependent pore-fluid densities have been addressed in this paper. In particular, a combination method of using theoretical and numerical approaches has been proposed to investigate the thermal structure of the continental lithosphere due to heat transfer from the continental mantle into upper crust of the Earth, which can be, in essence, attributed to a large-scale heat transfer problem in porous media with temperature-dependent pore-fluid densities.

In the proposed combination method, the theoretical approach is used to determine the thickness and the related thermal boundary conditions of the continental crust on the lithospheric scale, whereas the numerical approach is used to simulate the detailed structures and complicated geometries of the crust on the crustal scale. The main advantage in using the proposed combination method is that if the thermal structure in the crust is of the primary interest, the use of a numerical model on the crustal scale can result in a significant reduction in computer efforts. For this particular reason, analytical solutions for the relationship between the continental lithosphere thickness and the mantle conductive heat flux on the lithospheric scale have been mathematically derived in this study.

Three different kinds of lithosphere models, namely the conductive-only lithosphere, the conductive-and-advective lithosphere with constant pore-fluid density and the conductive-and-advective lithosphere with variable (i.e. temperature-dependent) pore-fluid density, are considered to derive such analytical solutions. The present analytical solution can be used to determine the thermal boundary condition, which is essential for the numerical modeling of ore body formation and mineralization on the crust or basin scale. From the ore body formation and mineralization points of view, the present analytical and numerical solutions have demonstrated that the conductive-and-advective lithosphere with variable pore-fluid density is the most favorite lithosphere because it may result in the thinnest lithosphere so that the temperature at the near surface of the crust can be hot enough to generate the shallow ore deposits there. In addition, the present analytical and numerical solutions also demonstrate that the upward throughflow (i.e. mantle mass flux) can have a significant effect on the thermal structure within the lithosphere.

For the purpose of considering the geometrical and material complexity of the lithosphere, the finite element formulation has been presented to deal with large-scale heat transfer problems with temperature-dependent fluid densities. After the present finite element formulation is well validated using several benchmark problems, it has been used to investigate how the heterogeneity of the material thermal conductivity can affect the temperature distribution within the lithosphere. The related numerical results have demonstrated that the emplacement of hot materials from the mantle may further reduce the thickness of the lithosphere.

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